Global singularization and failures of SCH

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Definition 1 F *is an* Easton function *if for all regular cardinals* κ, μ*:*

(i) If $\kappa < \mu$*, then* $F(\kappa) \leq F(\mu)$*;*
(ii) $\kappa < \mathrm{cf}(F(\kappa))$*.*

By results of W.B.Easton, if we assume GCH then every Easton function F is a *continuum function* ($\kappa \mapsto 2^\kappa$) on regular cardinals in some cofinality-preserving generic extension.

We might ask what additional assumptions are compatible with a given Easton function F (and hence with a realised continuum function). To formulate these assumptions we will need the following large cardinal.

Definition 2 *We say that κ is μ-strong (μ-hypermeasurable), where μ is a cardinal, if there is an embedding $j : V \to M$ such that $j(\kappa) > \mu$ and $H(\mu)$ is contained in M.*

Note that the property of being μ-strong is expressible in ZFC, claiming the existence of a direct limit of measure ultrapowers (the directed system of measures is called an *extender*).

Conjecture 3 *(GCH) Let F be an Easton function. For simplicity assume that for every κ, $F(\kappa)$ is smaller than the least Mahlo cardinal above κ. Then there is a cardinal preserving extension V^* where F is realised on V-regular cardinals and every $F(\kappa)$-strong cardinal in V (where $F(\kappa) > \kappa^+$) is a singular cardinal of cof ω in V^*. Hence these cardinals will fail SCH in V^* in the degree prescribed by F, while realising F on all V-regular cardinals.*

Remark. By results of M.Magidor, M.Gitik, and W.Mitchell, to have a failure of SCH at a singular strong limit κ with $2^\kappa = \mu$ requires (consistency-wise) (almost) a μ-strong cardinal. So our assumption in the Conjecture is in some sense necessary.

Note. For a strong limit singular κ, SCH at κ just says that GCH holds at κ: $2^\kappa = \kappa^+$.

The basic problem with obtaining the Conjecture is the following:

Unlike cardinals of cof ω failing SCH (at least by our current knowledge), the $F(\kappa)$-strong cardinals in the ground model have strong reflection properties. This severely limits functions F which can be realised by forcing, starting from these large cardinals.

Example. The simple Prikry forcing Prk(κ) (κ needs to be measurable in order to define Prk(κ)) adding a new ω-cofinal sequence to κ can be used to get κ to fail SCH, providing that κ was first a **measurable cardinal failing GCH**.

However, by reflection properties of measurable cardinals, this means that **GCH has to fail** on an unbounded set below κ. This limits the Easton functions F which can be realised.

Definition 4 *A condition in* Prk(κ) *is of the form* (s, A) *where* s *is a finite sequence in* κ *and* A *is a subset of* κ *which lies in some fixed normal* κ*-complete ultrafilter* U *on* κ*. We assume that* $\max(s) < \min(A)$*. We say that* (s, A) *is stronger than* (t, B)*,* $(s, A) \leq (t, B)$*, if* s *end-extends* t*,* $A \subseteq B$ *and* $s \setminus t \subseteq B$*. We say that* (s, A) *directly extends* (t, B)*,* $(s, A) \leq^* (t, B)$*, if* (s, A) *extends* (t, B) *and moreover* $s = t$*.*

Example. The extender based Prikry forcing $\mathrm{Prk}_E(\kappa, \mu)$ adds μ-many (where cf(μ) is at least κ^{++}) cofinal sequences of length ω without changing V_κ. Hence this forcing can be used to get κ fail SCH while **preserving GCH below** κ.

However, this forcing assumes that κ is roughly μ-strong in order to define $\mathrm{Prk}_E(\kappa, \mu)$ and that GCH holds sufficiently often below κ. This again restricts the permitted F's: for instance, an unbounded set of large cardinals must be preserved below κ (hence we cannot singularize all large cardinals below κ), and **GCH should hold** on an unbounded set below κ as well.

Prk(κ) and Prk$_E(\kappa, \mu)$ (and their variants) are the only forcings known so far which can be used to singularize cardinals in order to obtain failure of SCH. This means that with current techniques, the full version of Conjecture is beyond our reach.

But some fairly general results can be obtained.

Assume that we are interested just in Easton functions F which either preserve GCH at κ or fail it in the least way possible, i.e. for all regular κ, either $F(\kappa) = \kappa^+$ or $F(\kappa) = \kappa^{++}$. We call such F *toggle-like*.

Theorem 5 (Special case) *(GCH) Let F be a toggle-like Easton function. Then there is a cardinal-preserving extension V^* realising F on all V-regular cardinals such that cardinals in any fixed subclass of $F(\kappa) = \kappa^{++}$-strong cardinals of V are turned into singular cardinals of cof ω (and thus fail SCH).*

This shows that reflection properties of singular cardinals failing SCH with cof ω – if they exist – are not formulated in terms of GCH failing or holding below such cardinals.

Remark. Actually, we have cheated a little. There is a small (and probably erasable) side condition on F: for each measurable κ, $F(\kappa^+) = \kappa^{++}$. This condition is required by some technical aspects of the construction regarding $\mathsf{Prk}_E(\kappa, \kappa^{++})$.

Remark. This special case avoids the rather interesting case of κ with $F(\kappa)$ being a singular cardinal. Incorporation of such cases is possible, but the statement of the theorem gets more involved.

Sketch of proof.

Step 0. For a toggle-like F, if κ is $F(\kappa) = \kappa^{++}$-strong, then either there is a witnessing embedding j such that

$$F(\kappa) = j(F)(\kappa) \geq F(\kappa)$$

or there is a witnessing embedding j' such that

$$j'(F)(\kappa) = \kappa^{+}$$

The first case will reserved for Prk(κ) (GCH needs to fail often below κ), the other case for $\text{Prk}_E(\kappa, \kappa^{++})$ (GCH needs to hold often below κ).

Note that for a non-toggle like F, it is no longer true that for a $F(\kappa)$-strong κ one always finds either j or j' as above. This is the reason for lesser genearality where arbitrary F's are concerned.

Advanced note. The full version of the theorem deals with arbitrary F, but only $F(\kappa)$-strong κ's for which there is either j or j' as above are finally cofinalized. (There are also few technical side conditions.)

Step 1. We first realise F on all V-regular cardinals except cardinals κ from a special group of $F(\kappa)$-strong cardinals which we will call θ_E (E for "extender based Prikry"), where

$$\theta_E = \{\kappa \mid F(\kappa) > \kappa^+, (\exists j) j : V \to M, j(F)(\kappa) = \kappa^+\}$$

The forcing is an iteration of products of forcing along the Mahlo limits of Mahlo cardinals of V, combining the Sacks forcing and Cohen forcing.

This realises F everywhere except at elements in θ_E.

One needs to verify that

–Every $F(\kappa)$-strong κ with $j(F)(\kappa) \geq F(\kappa)$ remains a measurable cardinals with $2^\kappa = F(\kappa)$.

–Elements in θ_E remain sufficiently large to define $\mathsf{Prk}_E(\kappa, F(\kappa))$.

This requires the technique developed jointly with Sy D. Friedman (APAL, 154(3), 2008). The most interesting case is when $F(\kappa)$ is singular.

Step 2. We iterate with the Easton support the combination of Prk(κ) and Prk$_E(\kappa, F(\kappa))$ along some large cardinals of the first generic extension. This is a Prikry-style iteration which will

–Singularize measurable cardinals with $2^\kappa = F(\kappa)$, using Prk(κ)

–Realise F on elements of θ_E and simultaneously singularize $\kappa \in \theta_E$, using Prk$_E(\kappa, F(\kappa))$.

This tends to be very technical, especially at limit points of the construction.

Some technical details.

Question. Why not to first deal with $F(\kappa)$-strong cardinals over a ground model with GCH, and only then realise F elsewhere?

Forcing new subsets cofinally often below a cardinal κ failing SCH tends to collapse cardinals.

Example. Consider the following configuration. κ has cof ω, GCH holds below κ and $2^\kappa > \kappa^+$. Let $\langle \lambda_i \,|\, i < \omega \rangle$ be some regular cardinals cofinal in κ. Assume we want to add a single Cohen subset to each of λ_i.

Because the cofinality of κ is ω we do not have much choice as regards the support of the product/iteration of $\mathsf{Add}(\lambda_i, 1)$. Either finite support or full support.

Claim 6 *There are λ_i cofinal in κ such that both $\prod_{i\in\omega}^{\mathsf{FIN}} \mathsf{Add}(\lambda_i, 1)$ and $\prod_{i\in\omega}^{\mathsf{FULL}} \mathsf{Add}(\lambda_i, 1)$ collapse cardinals.*

Hint to proof. The more interesting one is the $\prod_{i\in\omega}^{\mathsf{FULL}} \mathsf{Add}(\lambda_i, 1)$. By Shelah theorem, there are λ_i cofinal in κ such that $\prod_{i\in\omega} \lambda_i/\mathsf{FIN}$ has true cofinality κ^+. This κ^+-sequence can be used to argue that $2^\kappa = |\prod_{i\in\omega} \lambda_i|$ is collapsed to κ^+ (generically, every element of $\prod_{i\in\omega} \lambda_i$ can be coded using an element in the true-cofinality sequence).

In **Step 1**, the taks of preservation of large cardinals is achieved by lifting the original witnessing embedding $j : V \to M$ to the generic extension $j^* : V[G] \to M[j^*(G)]$, where $j^* \restriction V = j$.

In Prikry iteration in **Step 2**, it is not possible to lift the whole j, but individual measures are lifted (but a normal measure in V may be necessarily extended into a non-normal measure). This uses the properties of Prikry-type forcings.

We say that a forcing notion $(P, \leq, \leq^*)$ with $\leq^* \subseteq \leq$ is Prikry-type if for every sentence σ and every $p \in P$ there is some $q \leq^* p$ deciding σ. $\leq^*$ is called the *direct-extension relation*. $\leq^*$ is typically more closed than $\leq$. The basic example is Prk(κ), where $\leq^*$ is κ-closed, while $\leq$ is not even ω_1-closed.

It can be shown that a correctly defined iteration of Prikry-type forcing notions is also Prikry-type.

Assume for simplicity that $\mathbb{R}_\kappa$ is an iteration of just Prk(α) for unboundedly many $\alpha < \kappa$. Typically, to define Prk(κ) in $\mathbb{R}_\kappa$ we need to make sure that κ remains measurable in $\mathbb{R}_\kappa$. Let H_κ be a $\mathbb{R}_\kappa$-generic. For $\dot{X}$ a $\mathbb{R}_\kappa$-name for a subset of κ we define a measure U in $V[H_\kappa]$ as follows

$$X \in U \text{ iff } \exists p \in H_\kappa \exists q \leq^* 1_{j(\mathbb{R}_\kappa)\setminus\mathbb{R}_\kappa} p^\frown q \Vdash \kappa \in j(\dot{X}),$$

where j witnesses for measurability (or other largeness) of κ in V.

The above definition of a measure U works fine providing that all direct extensions of $1_{j(\mathbb{R}_\kappa)\setminus\mathbb{R}_\kappa}$ are compatible.

This is true only in a limited setting: when $\mathbb{R}_\kappa$ has full support, and when all non-trivial forcings in $\mathbb{R}_\kappa$ have themselves this property (such as $\mathsf{Prk}(\alpha)$). For instance, $\mathbb{R}_\kappa$ cannot contain $\mathsf{Prk}_E(\kappa, F(\kappa))$.

Our iteration $\mathbb{R}$ typically contains $\mathsf{Prk}_E(\kappa, \mu)$. How to deal with this situation:

–The witnessing $j : V \to M$ in the ground needs to be of a special kind (even for Step 1), the so called extender embedding:

$$M = \{j(f)(\alpha) \mid f : \kappa \to V, \alpha < F(\kappa)\}$$

–To compatibly hit some weakly-dense open sets, we may argue only f's from κ to $H(\kappa^+)$ need to be considered in M. Using GCH in V, all weakly-dense open sets can be grouped into κ^+-many segments. Since M is closed under κ-sequences, this will be enough.

More or less the end. (More in the lecture).

Preprint will be available soon on my webpage.